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RESOURCE DEPLETION UNDER TECHNOLOGICAL UNCERTAINTY[1]

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The purpose of this paper is to study the effect of uncertainty in the arrival date of a new technology on the rate of depletion of an exhaustible natural resource. It is shown that under a large class of circumstances uncertainty leads to a faster initial depletion rate if the initial resource stock is small and to greater conservation if it is large. A particular kind of certainty equivalence result is proved and the results of the paper are used to comment on possible interpretations of certain historical episodes of resource exhaustion.

1. INTRODUCTION

IN THIS PAPER we shall study the effect of uncertainty on the rate of depletion of an exhaustible natural resource. The particular kind of uncertainty we shall postulate here arises from a random arrival date of a new technology. The key feature of the new technology we postulate (e.g. controlled nuclear fusion for energy generation) is that it will enable society to produce a perfect substitute for the resource at a known constant (unit) cost.

We might conjecture that since uncertainty makes the return to holding the stock (of the resource) riskier, it encourages depletion in a market economy. But there is another argument which runs the other way, which is that "prudence"—i.e. concern for future generations, lest they be left with an inadequate supply of the resource—will dictate a desire for greater conservation. In fact we establish below that the effect of uncertainty depends on the *size* of the stock; a result which ought to come as no surprise. What is, however, somewhat of a surprise is that for an important class of cases, viz. when demand is not too inelastic at high enough prices, uncertainty leads to a faster depletion rate initially if the stock is *small* and to a greater conservation if it is *large*. We shall provide a heuristic explanation of this somewhat counter-intuitive result.

A related question is whether there exists a certainty equivalent date of invention of the new technology. This is of interest especially because numerical calculations of the rate at which resources ought to be depleted have usually been undertaken on the assumption that a new technology will be invented at some *known* future date (see, e.g., Nordhaus [**9**]). We shall show in what follows that there is no certainty equivalent date of invention. The magnitude of the error committed in replacing the random variable by some statistic (e.g. the mean) could indeed be great.

The model that we explore in this paper is a very special one. It hypothesizes a single-grade resource and a single invention. The resource stock is known with

[1] This is an extended version of the first part of Dasgupta and Stiglitz [**3**], research towards which was financed partially by the National Science Foundation Grant No. SOC74-22182 when the authors were both at Stanford University. This present version was prepared while Dasgupta was a Visiting Professor at the School of International Studies, Jawaharlal Nehru University and the Delhi School of Economics, and Stiglitz was Oskar Morgenstern Distinguished Research Fellow at Mathematica and Visiting Professor at the Institute of Advanced Studies, Princeton, during the autumn of 1978. We are most grateful to Paul David, Richard Gilbert, and Stephen Nickell for helpful discussions.

certainty, but the date of invention of the new technology is random. R and D is not a decision variable in this paper.[2] Nor is there any uncertainty about the characteristics of the new technology. These are extensions that can be incorporated in a straightforward manner. But the model postulated here captures in a sharp manner the problem of the transition from an exhaustible to an inexhaustible resource.[3]

The model developed here should also provide a cautionary note for the interpretation of certain historical episodes. During the 16th century, deforestation was followed by the introduction of coal as a source of fuel in England. This is usually interpreted by the thesis that deforestation *caused* the introduction of coal: the rise in the price of timber led to innovations in coal extraction.[4] The model analyzed in this paper provides an alternative interpretation, which is that the discovery of superior methods of coal extraction meant that it was uneconomical to rely on timber as a fuel. Existing stands of timber were then treated as an exhaustible natural resource, rather than a renewable one. The model suggests that prior to deforestation the price of timber would have risen roughly to the production cost of an equivalent amount of coal. The thesis is that invention of better methods of coal extraction, or, indeed, knowledge that such an invention was likely to become available in the near future *preceeded* the exhaustion of timber and was the *cause* of what happened afterwards. But the model does indeed predict that *innovation* would follow the exhaustion of timber.[5]

2. THE BASIC ARBITRAGE EQUATION

We begin our analysis by extending to the context of uncertainty the fundamental arbitrage equation for natural resources. For simplicity we shall assume in what follows that extraction cost for the exhaustible resource is nil. Let p_t denote the spot price of the resource at t. Consider the short time interval $(t, t+\theta)$. If there is no chance of a substitute product being invented during this interval the return to holding the resource is

$$(p_{t+\theta} - p_t)/p_t,$$

and in equilibrium this must equal the return $(r_t\theta)$ for holding a bond for this same length of time (where r_t is the rate of return on the bond). Thus

$$(p_{t+\theta} - p_t)/p_t = r_t\theta,$$

[2] This problem has been studied in Dasgupta, Heal, and Majumdar [**4**] and Kamien and Schwartz [**5**].

[3] For a general discussion of this problem, see Koopmans [**6**].

[4] For a summary of this thesis, see Nef [**8**].

[5] We are most grateful to Paul David for pointing out this possible connection to us. E. Steinmuller [**11**], in an unpublished paper, has recently developed this argument in detail and presented historical evidence in its support.

which, on taking $\theta \to 0$, yields

(1) $\dot{p}_t/p_t = r_t.$

Equation (1) is, of course, well known.

Now assume that during $(t, t+\theta)$ there is a probability $\lambda_t\theta$ of an event occurring. For the rest of this paper we shall interpret the event as being the *invention* of a product which is a perfect substitute for the resource and whose production cost is known in advance.[6] If the invention does occur, the economy enters a new regime. Assume that in this event the competitive price becomes $\hat{p}_t$. We shall refer to $\hat{p}_t$ as the *fall-back* price of the resource. For the model at hand the only *endogenous* factor which determines $\hat{p}_t$ is the resource stock at t, S_t. Thus write $\hat{p}_t = \hat{p}(S_t)$. If speculators are risk neutral then in dynamic equilibrium one has

$$\lambda_t\theta\hat{p}(S_t) + (1-\lambda_t\theta)(p_t + dp_t) = (1 + r_t\theta)p_t,$$

which, on taking limits, as $\theta \to 0$, yields the arbitrage condition

(2) $\dot{p}_t/p_t = r_t + \lambda_t(1 - \hat{p}(S_t)/p_t).$[7]

Certain special cases of (2) may now be mentioned. If either $\lambda_t = 0$ (i.e. there is no chance that the invention will be made at t) or if $\hat{p}(S_t) = p_t$ (i.e. the invention has no bearing on the market for the resource), then (2) reduces to (1). But as we are supposing that the invention is that of a substitute product it is simple to confirm that $\hat{p}(S_t) < p_t$ (see Section 4). Thus (2) implies that

(3) $r_t \leqslant \dot{p}_t/p_t \leqslant r_t + \lambda_t.$

It is only when $\hat{p}(S_t) = 0$ (i.e. the invention renders the existing stock worthless) that $\dot{p}_t/p_t = r_t + \lambda_t$.[8]

We now introduce the demand side of the model. The analysis that follows is strictly partial equilibrium in nature. The marginal utility of income is assumed constant. Let the market demand curve for the flow of services provided by the resource, x, be given by a continuously differentiable function $f(x)$. We are supposing for simplicity of exposition that demand does not shift. We take it that $f'(x) < 0$ and, to avoid corner problems, that $\lim_{x\to 0} f(x) = \infty$, and that $\lim_{x\to\infty} f(x) = 0$. Gross consumer surplus at the rate of flow x is therefore $u(x) \equiv \int_0^x f(z)\,dz$.[9]

Having obtained the arbitrage condition and described demand conditions, we proceed to characterize extraction paths along intertemporal competitive equilibria. For ease of exposition we take it that $r_t = r > 0$. In what follows we shall

[6] Which is to say that R and D activity, which is going on in the background, is highly goal oriented. The technology postulated here is often called a back-stop technology.

[7] Equation (2) is very general and will be valid even when one contemplates an entire sequence of possible events over time. It represents the equilibrium condition at t; so, λ_t denotes the probability rate that a specific event occurs at t conditional on its not having occurred earlier.

[8] This special case was discussed in Dasgupta and Heal [**2**].

[9] For reasons that should be familiar we shall need to suppose as well that

$$\infty > \lim_{x\to 0}\{-f'(x)x/f(x)\} = \eta > 0.$$

always suppose that the sequence of momentary equilibria sustains an intertemporal competitive equilibrium, which is to say, we suppose that "market forces" ensure that the "transversality condition" is satisfied. One way of justifying this is to postulate the existence of a complete set of Arrow-Debreu markets. Alternatively (and this is the route we pursue here), one may wish to assume straightaway that we are considering a planned economy in which the planner is concerned with maximizing the expected present value of net social surplus.[10]

3. COMPETITIVE EXTRACTION WHEN INVENTION DATE IS PERFECTLY FORESEEN

Let x_t denote the flow of the resource at t. At $t = 0$ the economy is provided with an initial stock S_0. It is known that the invention will consist of a perfect substitute, whose unit cost of production is $\bar{p} > 0$ (this is the only invention that is envisaged). Let y_t denote the flow of the substitute product at t. We take it that it is known that the invention will be made at date $T^*(\geq 0)$. Our task is to describe the competitive equilibrium outcome. Formally, we obtain the characteristics by solving the following planning problem:

$$\text{(4)} \qquad \underset{(x_t, y_t)}{\text{maximize}} \int_0^\infty e^{-rt}(u(x_t + y_t) - \bar{p}y_t))\, dt,$$

subject to the constraints

$$S_t = S_0 - \int_0^t x_\tau \, d\tau,$$

$$x_t,\ y_t,\ S_t \geq 0 \qquad \text{for all } t \geq 0, \qquad \text{and} \qquad y_t = 0 \qquad \text{for } 0 \leq t < T^*.$$

Given the assumptions that have been made about $f(x)$ it is simple to confirm that a solution to (4) exists and that the solution is unique (see Appendix 1). The general characteristics of the solution are routine to obtain and we present proofs in Appendix 1. The salient features can then be stated in the form of the following proposition.

PROPOSITION 1: *There exists a stock level $S^* = S^*(T^*)$ (with $S^*(0) = 0$ and $dS^*/dT^* > 0$), such that if $S_0 > S^*$ the competitive path consists of three phases: (a) pre-invention, (b) post-invention and pre-innovation, and (c) post-innovation. Throughout, the price is continuous and during phases (a) and (b) the resource price obeys equation (1) with $p_t < \bar{p}$. During phase (b) the new technology is held in abeyance. The initial price of the resource has the property that at the date of innovation $\hat{T}(\hat{T} > T^*)$ the entire resource stock is exhausted and the economy enters phase (c), along which the new technology is in use and the commodity is sold at the price $\bar{p}$ (see Figure 1). If $S_0 < S^*$, phase (b) does not occur and the economy moves directly from phase (a) to phase (c) and there is a discontinuous fall in price at the*

[10] Here we are appealing to the Fundamental Theorem of Welfare Economics. We suppose of course that a maximum exists. We shall explore this question in the Appendices.

date of invention T^*, which is also the date of resource exhaustion and technological innovation (see Figure 2). If $S_0 = S^*$, phase (b) does not occur, but the price is throughout continuous (Figure 3).

Proposition 1 is, of course, eminently congenial to intuition. Since by assumption $\lim_{x\to 0} f(x) = \infty$, the economy carries a positive inventory of the resource so long as the invention has not been made. Thus, if T^* is far away in the future (as in Figure 2) the initial price, p_0, of the resource is chosen sufficiently high so as not to allow the economy to run out of the resource stock prior to T^*. Nor is it chosen so high as to allow the economy to carry an inventory beyond T^*, for then consumption in the early stages will be too low. So long as stocks last the spot price obeys equation (1), and the flow market clears at each instant via the demand function $f(x)$. Therefore, the initial price is so chosen that the last unit of the resource is used at the date of invention, and the economy makes an immediate

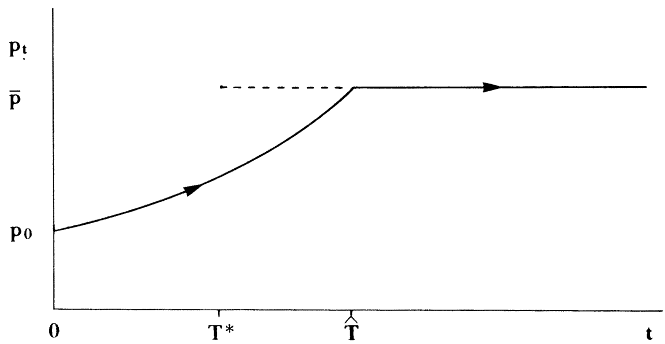

FIGURE 1.—Price movement when $S_0 > S^*(T^*)$. T^* is date of invention and $\hat{T}$ is date of innovation and resource exhaustion.

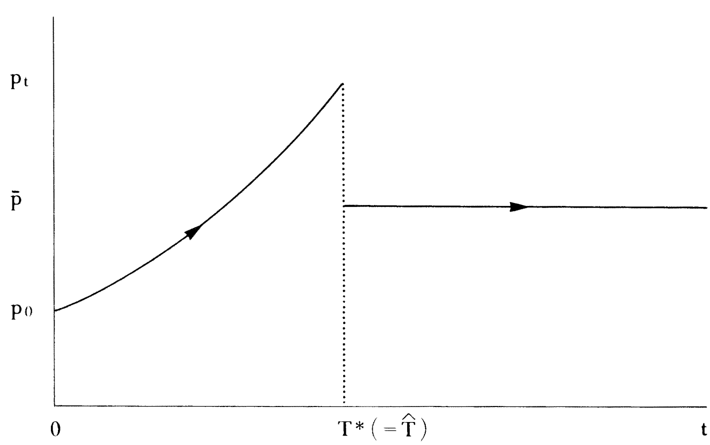

FIGURE 2.—Price movement when $S_0 < S^*$.

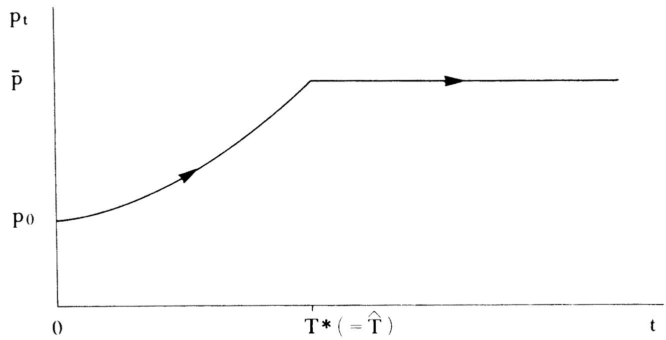

FIGURE 3.—Price movement when $S_0 = S^*$.

transition to the new technology. However, if the invention date is near at hand (relative to the inherited stock), as in Figure 1, matters are different. To see this sharply suppose $T^* = 0$. The trade-off to be contemplated by the planner is between the use of a cheap resource (zero extraction cost) with a finite base, and an expensive resource (production cost $\bar{p}$), with an infinite base. Concurrent exploitation is patently sub-optimal and in fact one wants to delay innovation and thus save on production costs. The question arises as to when the economy ought to innovate. Now as long as stocks are allowed to last equation (1) holds and the flow of consumption falls along the demand curve as the resource price rises. For the case at hand the initial price p_0 can be so chosen that the market clearing price (and, therefore, marginal utility) is a continuous function of time. It is then almost immediate that p_0 is so chosen that at the date price rises to the level $\bar{p}$ via (1), the entire stock is exhausted. Let $\hat{T}$ be this date. Innovation occurs at $\hat{T}$. But then this also explains the first and third parts of Proposition 1 (Figures 1 and 3). For if society plans to innovate at $\hat{T}$ when the innovation is already available it will choose to innovate at $\hat{T}$ even if the date of invention, T^*, is in the future, so long as $T^* \leq \hat{T}$.

In fact Proposition 1 highlights the fact that there is a precise sense in which one can have an invention too early. In the case when $S_0 > S^*(T^*)$, access to a backstop technology is not an argument for introducing it (phase (b)). Indeed, under competitive conditions the backstop will be priced out of the market so long as there is some resource stock, since resource owners will undercut the competitive producers.[11] It is then clear that if T^* were endogenous, then in the absence of uncertainty in R and D the optimum date of invention would not be earlier than the date, say T, for which $S_0 = S^*(T)$;[12] i.e. if T^* is endogenous, then if it is possible to avoid phase (b), it should be avoided.

[11] The argument holds even if there are extraction costs, so long as the marginal cost of extraction falls shoft of $\bar{p}$.

[12] This supposes, as is plausible, that increasing the speed of research has positive costs.

4. COMPETITIVE EXTRACTION WHEN INVENTION DATE IS UNCERTAIN

We turn now to the central case to be dealt with in this paper, the case where the invention date is uncertain. As earlier, let λ_t denote the probability rate of the invention occurring at t conditional on it not having occurred earlier. In what follows we suppose $\lambda_t(t \geq 0)$ is continuous at all $t \geq 0$. From the vantage point of $t=0$ let Π_t be the probability rate of the invention being made at t. Define $\Omega_t = \int_0^t \Pi_\tau \, d\tau$. Then clearly $\lambda_t = \Pi_t/(1-\Omega_t)$. Among other things the supposition that λ_t is continuous at all $t \geq 0$ implies that $\Omega_t < 1$ for all $t \geq 0$. But we need not rule out for the moment that $\Omega_\infty < 1$.

In analyzing the problem at hand we make use of a dynamic programming argument. Let t be the date of invention along a sample path, and let S_t be the resource stock remaining. The competitive outcome subsequent to t is, of course, given by the first part of Proposition 1 (see Figure 1).[13] Let x_τ be the resource flow and y_τ the output flow of the substitute product at $\tau(\tau \geq t)$. Now define

$$V(S_t) \equiv \max_{(x_\tau, y_\tau)} \int_t^\infty e^{-r(\tau-t)}[u(x_\tau + y_\tau) - \bar{p}y_\tau]\, d\tau, \tag{5}$$

subject to the constraints

$$\int_t^\infty x_\tau \, d\tau \leq S_t, \qquad \text{and} \qquad x_\tau, y_\tau \geq 0.$$

Next, define $\hat{p}(S) \equiv V'(S)$. This is clearly the initial price for the competitive equilibrium path described in the first part of Proposition 1, and is in fact what we have called the fall-back price in Section 2.

It remains for us to obtain S_t along the intertemporal competitive equilibrium when the date of invention is uncertain. This is obtained by solving the optimization problem at $t=0$, which is to

$$\underset{(x_t)}{\text{maximize}} \int_0^\infty e^{-rt}[u(x_t)(1-\Omega_t) + \Pi_t V(S_t)]\, dt, \tag{6}$$

subject to the constraints

$$S_t = S_0 - \int_0^t x_\tau \, d\tau, \qquad \text{and} \qquad S_t, x_t \geq 0, \qquad \text{all } t \geq 0.$$

In what follows we suppose that (6) has a solution and that it is unique. The existence problem is explored in Appendix 2.

Let p_t denote the spot (shadow) price of the resource at t emerging from (6). In Appendix 2 we shall note that the Euler-Lagrange equation associated with (6) is in fact equation (2). The planner pursues the solution of (6) so long as the invention has not been made. Therefore p_t is the spot shadow price of the resource

[13] The planner merely places himself at the vantage point of date t, regards S_t as the inherited stock, and solves problem (4) by noting that the substitute source is at hand.

at t conditional on the invention not having occurred by t. In particular, define

$$W(S_t) \equiv \max_{(x_t)} \int_t^{\infty} e^{-r(\tau-t)}[u(x_\tau)(1-\Omega_\tau)+\Pi_\tau V(S_\tau)]\,d\tau/(1-\Omega_t), \tag{7}$$

subject to the constraints

$$S_\tau = S_t - \int_t^\tau x_q\,dq; \qquad \text{and} \qquad S_\tau, x_\tau \geqslant 0 \qquad \text{all } \tau \geqslant t.$$

Clearly then $p_t = W'(S_t)$. Moreover, as the problem is one of concave programming $V'(S) < W'(S)$ for all $S \geqslant 0$. This implies that $p_t > \hat{p}(S_t)$, and from (5) that $\bar{p} > \hat{p}(S_t)$. That $p_t > \hat{p}(S_t)$ can be checked as well by the following heuristic consideration. For any level of stock, S_t, $p(S_t)(= W'(S_t))$ is the (shadow) price for the resource at t conditional on the invention not having arrived by then, whereas $\hat{p}(S_t)$ is the (shadow) price of the resource at t computed on the assumption that the invention *has* occurred by then. The resource is clearly more valuable at the margin in the former case. Now, the fact that $\bar{p} > \hat{p}(S_t)$ for all $S_t > 0$ follows from the first part of Proposition 1 (see Figure 1). Next, note that since both $V(S)$ and $W(S)$ are strictly concave, $\hat{p}'(S), p'(S) < 0$. Finally, as we have supposed that $\lim_{x\to 0} f(x) = \infty$, it is clear from the first part of Proposition 1 that $\lim_{S\to 0} \hat{p}(S) = \bar{p}$, and from (7) that $p_t \to \infty$ if $S_t \to 0$.

We may now characterize the competitive path in the form of Proposition 2:

PROPOSITION 2: *A sample path consists of three phases:* (*a*) *pre-invention,* (*b*) *post-invention and pre-innovation, and* (*c*) *post-innovation.*[14] *During phase* (*a*) *the resource price obeys equation* (2), *and the initial price is so chosen that $S_t > 0$ for all t and $\lim_{t\to\infty} S_t = 0$. At the date of invention the price falls discontinuously to the fallback level, which is less than $\bar{p}$, and the economy enters phase* (*b*). *During this second phase the resource price is less than $\bar{p}$ and it satisfies equation* (1). *The invention is kept in abeyance. The fallback price has the property that at the date the resource price reaches $\bar{p}$ the entire stock is exhausted and the economy enters phase* (*c*), *along which the substitute product is produced and sold at the price $\bar{p}$ (see Figure* 4).

The proof of this proposition is routine and is presented in Appendix 2 for completeness. But the point to note is that the economy moves along the price trajectory ABC in Figure 4, so long as the substitute technology is unavailable. Since by assumption $\lim_{x\to 0} f(x) = \infty$, the economy carries a positive inventory of the resource so long as the invention has not been made. The spot price meanwhile obeys equation (2), and the flow market clears at each date via the demand function $f(x)$. The initial price, p_0^u (Figure 4), gets so chosen that $\lim_{t\to\infty} S_t = 0$. If the initial price were to be chosen at a higher level the myopic rule (equation (2)) would lead the economy along a dynamically inefficient path, with $\lim_{t\to\infty} S_t > 0$. If it were lower, the myopic rule would result in resource exhaustion in finite time,

[14] For phases (b) and (c) to occur with probability one we need to suppose that $\Omega_\infty = 1$.

which is to say that there would be a positive probability for the economy being caught with no resource and no substitute. Thus the price trajectory ABC in Figure 4 is the one the economy follows so long as the substitute technology is not at hand. However, by assumption, $\Omega_\infty > 0$. Thus let T_1 be the date of invention along a sample path. At T_1 the economy finds itself with a new technology and a left over stock from the earlier era. It now possesses a cheap resource (zero extraction cost) with a finite base and an expensive resource (unit cost of production $\bar{p}$) with an infinite base. The first part of Proposition 1 is now of relevance. What Proposition 2 says is that at T_1 the economy replans by following the first part of Proposition 1.

5. THE EFFECT OF UNCERTAINTY ON RESOURCE EXTRACTION

We are now concerned with analyzing the nature of the bias in the *initial* rates of extraction that results if the uncertain date of invention is replaced by its expected value. Equivalently, we can study the bias in the initial market price. Towards this we simplify and suppose that the stochastic process is a Poisson one. Thus $\lambda_t = \lambda > 0$. This will enable us to conduct the analysis using phase diagrams.

In what follows we denote by p_0^u the initial competitive price of the resource when the date of invention is random (see Figure 4) and by p_0 the initial competitive price if it is known with certainty that the invention will be made at the

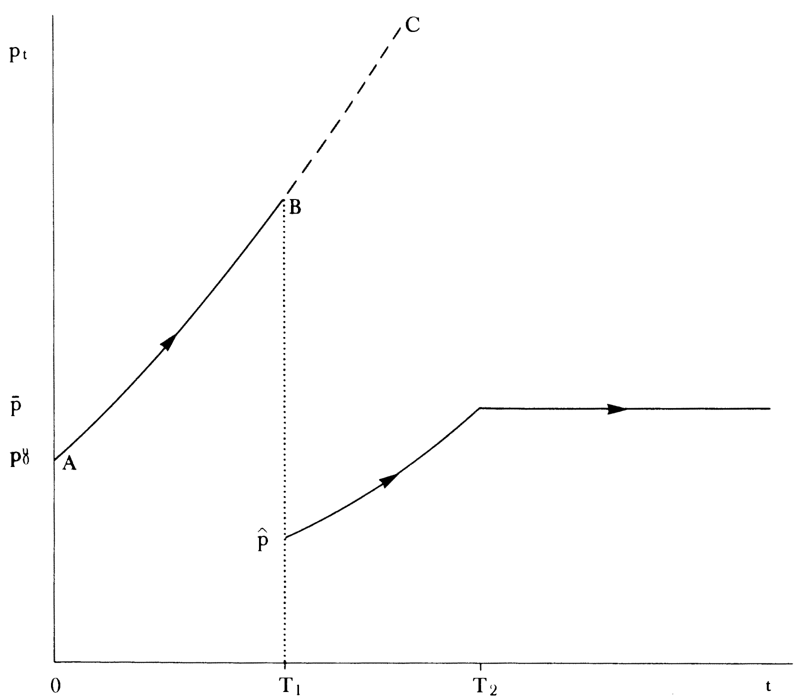

FIGURE 4.—Sample path along which invention occurs at T_1. During $(0, T_1)$ equation (2) holds and during (T_1, T_2) equation (1) holds. T_2 is the date of innovation at which the resource is exhausted.

expected date $T^*(=1/\lambda)$ (see Figures 1–3). Our task is to compare p_0^u and p_0. If $p_0^u > p_0$, uncertainty in the date of invention of a substitute product provides a planner with an argument for conservation at initial dates, and for profligacy if $p_0^u < p_0$. Now we know in advance that each of these prices depends on the initial stock S_0. Our task then is to study the characteristics of the functions $p_0^u(S_0)$ and $p_0(S_0)$.

Let $D(p) = f^{-1}(p)$ be the market demand function. Then we have noted that the competitive path under uncertainty (phase (a) in Proposition 2) is characterized by equation (2) and the condition

$$\text{(8)} \qquad \dot{S}_t = -D(p_t),$$

where $S_t > 0$ and $\lim_{t\to\infty} S_t = 0$.

Using equations (2) and (8) one obtains immediately that

$$\text{(9)} \qquad \frac{dp_0^u}{dS_0} = -\frac{(rp_0^u + \lambda p_0^u(-\hat{p}(S_0)/p_0^u))}{D(p_0^u)}.$$

(Recall that the fallback price $\hat{p}(S)$ is the initial price for the first part of Proposition 1 (Figure 1).)

Using equation (9) the locus describing the initial price p_0^u as a function of the initial stock, S_0, is drawn in Figure 5, as AA'.[15] Indeed, AA' is the locus that the economy follows as a solution to (6). For, since the optimum plan is intertemporally consistent, AA' is the phase path the economy will wish to follow.

We now turn to the case where it is known with certainty that the invention will occur at the expected date $T^* = 1/\lambda$. Suppose first $S_0 < S^*(1/\lambda)$ (see Proposition 1). Then we know from Proposition 1 that the competitive path satisfies the condition

$$\text{(10)} \qquad \int_0^{1/\lambda} D(p_0 e^{rt})\, dt = S_0,$$

where p_0 is the initial price (see Figure 2).[16] Let $\eta(p)$ be the absolute value of the price elasticity of demand (i.e. $\eta(p) = -pD'(p)/D(p)$). Write

$$\bar{\eta} \equiv \int_0^{1/\lambda} \eta(p_t) D(p_t)\, dt/S_0$$

as the weighted average of the elasticity of demand along the competitive program, during the pre-invention era. It then follows on differentiating equation (10) that

$$\text{(11)} \qquad \frac{dp_0}{dS_0} = -p_0/\bar{\eta}S_0 \qquad \left(\text{for } S_0 < S^*\left(\frac{1}{\lambda}\right)\right).$$[17]

[15] From problem (6) it is immediate that, $dp_0^u/dS_0 < 0$. Moreover, since $p_0^u \to \infty$ as $S_0 \to 0$, $dp_0^u/dS_0 \to -\infty$ as $S_0 \to 0$. Furthermore, $p_0^u \to 0$ as $S_0 \to \infty$, and $dp_0^u/dS_0 \to 0$ as $S_0 \to \infty$.

[16] The spot price satisfies (1) so long as the stock lasts, and since $S_0 < S^*(1/\lambda)$, the initial price is so chosen that innovation occurs at the date of invention, so that the stock is exhausted precisely at date $1/\lambda$. Furthermore $p_0 e^{r/\lambda} > \bar{p}$ (Figure 2).

[17] Since by assumption $\lim_{x\to 0} f(x) = \infty$, we know that $p_0 \to \infty$ as $S_0 \to 0$. Since also by hypothesis $\infty > \lim_{p\to\infty} \eta(p) > 0$, $\bar{\eta}$ exists if $S_0 \to 0$, and so from (11) we obtain that $dp_0/dS_0 \to -\infty$ as $S_0 \to 0$.

Consider now the case where $S_0 > S^*(1/\lambda)$. From Proposition 1 we know that the date of innovation, $\hat{T} > 1/\lambda$ (Figure 1). In particular we know that $\hat{T}$ and p_0 are related by the conditions

$$(12) \qquad \int_0^{\hat{T}} D(p_0 e^{rt})\, dt = S_0$$

and

$$(13) \qquad p_0 e^{r\hat{T}} = \bar{p}.$$

From equations (12) and (13) it then follows that

$$(14) \qquad dp_0/dS_0 = -p_0 \Big/ \left(\bar{\bar{\eta}} S_0 + \frac{D(\bar{p})}{r} \right) \qquad (\text{for } S_0 > S^*(1/\lambda)),$$

where $\bar{\bar{\eta}} \equiv \int_0^{\hat{T}} \eta(p_t) D(p_t)\, dt / S_0$. Thus while $p_0(S_0)$ is a continuous function, equations (11) and (14) tell us that the function has a kink at $S_0 = S^*(1/\lambda)$. $p_0(S_0)$ is drawn in Figure 5 as the curve $BB'B''$, with the kink at B'.[18] In this figure the dotted curves, such as $B'D$ and MN indicate the *actual* movement of the price and the stock for different values of S_0 in the case where it is known that invention will occur at $T^* = 1/\lambda$. Thus if $S_0 > S^*(1/\lambda)$, the economy will follow the phase trajectory $B''B'D$ (see also Figure 1). Indeed, $B''B'D$ represents the fallback price as a function of the stock, $\hat{p}(S)$ (first part of Proposition 1). If, on the other hand, as

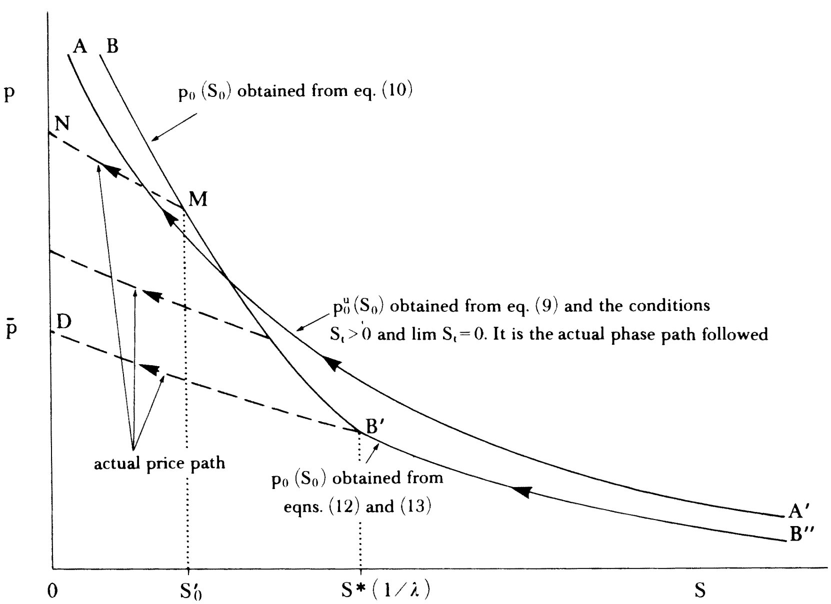

FIGURE 5

[18] We know that $p_0 \to 0$ if $S_0 \to \infty$ (see (4)). Thus from (14) we note that $dp_0/dS_0 \to 0$ if $S_0 \to \infty$.

in Figure 5, the initial stock is $S_0'(<S^*(1/\lambda))$, the initial price will be that associated with the point M, and the economy will follow the phase trajectory MN (the second part of Proposition 1), with the price at the date of invention exceeding $\bar{p}$, as at the point N (see Figure 2). An obvious further point to observe is that the locus $B'D$ lies entirely below $B'B$. This feature will be useful to recall subsequently.

In order to analyze the effect of uncertainty on the initial price we need to study the relative positions of the curves $BB'B''$ and AA' in Figure 5. We begin by demonstrating first that AA' must lie entirely *above* $B''B'D$. To see this, suppose $S_0 \leq S^*(1/\lambda)$. Then $B'D$ represents the fallback price associated with S_0. That is, $B'D$ represents $\hat{p}(S_0)$ for $S_0 \leq S^*(1/\lambda)$. Now suppose $p_0^u(S_0) = \hat{p}(S_0)$ for some $S_0 \leq S^*(1/\lambda)$. Then equation (9) reduces to $dp_0^u/dS_0 = -rp_0^u/D(p_0^u)$ at this S_0. But the slope of $B'D$ is $-r\hat{p}(S_0)/D(\hat{p}(S_0))$. Therefore $B'D$ and AA' have the same slope at this value of S_0 if we suppose that $p_0^u(S_0) = \hat{p}(S_0)$. It follows that $B'D$ and AA' cannot cross at this value of S_0. A fortiori they cannot cross at any $S_0 \leq S^*(1/\lambda)$.

We next show that AA' cannot cross $B'B''$ either. For suppose $S_0 \geq S^*(1/\lambda)$. Then note that

$$D(p_0) - D(\bar{p}) = -\int_0^{\hat{T}} \dot{D}\, dt = -\int_0^{\hat{T}} D'(p_t)\dot{p}_t\, dt = r\int_0^{\hat{T}} \eta(p_t)D(p_t)\, dt$$
$$= rS_0\bar{\bar{\eta}}.$$

Using this in equation (14) yields $dp_0/dS_0 = -rp_0/D(p_0)$. Again assume that $p_0^u = p_0(=\hat{p}(S_0))$, for some $S_0 > S^*(1/\lambda)$. Then equation (9) reduces to $dp_0^u/dS_0 = -rp_0/D(p_0)$. Therefore, the hypothesis that $p_0^u = p_0$ for some $S_0 > S^*(1/\lambda)$ entails that AA' and $B'B''$ have the same slope at this value of S_0. It follows that AA' and $B'B''$ cannot cross at this S_0. A fortiori they cannot cross at any $S_0 > S^*(1/\lambda)$. Therefore AA' must lie either entirely below $B''BD$ or entirely above it. Since along the phase path $B''B'D$ the stock is exhausted in finite time it must be exhausted in finite time along any phase path which is bounded above by it. But since the solution of (6) is intertemporally consistent for the Poisson case, $A'A$ is not merely the initial price as a function of the initial stock when the date of invention is random; it is also the locus the economy actually follows conditional on the invention not having been made. Now we have also noted that the solution to (6) has the economy carrying a positive inventory so long as the invention has not occurred. Therefore $A'A$ cannot lie entirely below $B''B'D$. We have therefore proved that $A'A$ must be entirely above $B''B'D$. We can now relate Figure 4 to Figure 5. Given an initial stock the competitive economy moves along the phase path $A'A$ so long as the invention has not occurred. The instant the invention is made the economy jumps down to the phase path $B''B'D$ at the point vertically below $A'A$ and moves along $B''B'D$ until the resource is exhausted at D.[19] Now, the fact that $A'A$ lies entirely above $B''B'$ implies that for $S_0 > S^*(1/\lambda)$ the initial

[19] Thus, while the claim preceding inequality (3) holds under more general conditions, we have in fact provided a proof for the case where $\lambda_t = \lambda > 0$, that $p_t > \hat{p}(S_t)$.

price is lower when the date of invention is perfectly predictable (at $T^* = 1/\lambda$) than when it is uncertain. Uncertainty results in greater conservation.

The question arises whether $A'A$ crosses $B'B$ or whether it lies entirely above it. We demonstrate first that if demand at large enough prices is not too inelastic then $A'A$ must cross $B'B$ at least once and in particular, for small enough a value of the initial stock $A'A$ lies below BB', i.e. that $p_0^u(S_0) < p_0(S_0)$ when $S_0 \to 0$, and therefore that uncertainty results in profligacy. To confirm this we use l'Hospital's rule on equations (9) and (11) to note that

$$(15) \qquad \lim_{S_0 \to 0} (p_0/p_0^u) = \lim_{S_0 \to 0} p_0 D(p_0^u)/\bar{\eta} S_0 (r p_0^u + \lambda p_0^u (1 - \hat{p}(S_0)/p_0^u)).$$

But $\hat{p}(S_0) \to \bar{p}$ and $p_0^u \to \infty$ as $S_0 \to 0$. Therefore (15) reduces to

$$(16) \qquad \lim_{S_0 \to 0} \frac{D(p_0^u)}{S_0} = (r + \lambda) \lim_{S_0 \to 0} \bar{\eta}.$$

Equation (16) relates p_0^u to the initial stock S_0, when S_0 is "small." We now need to find the functional relationship between p_0 and S_0 when S_0 is "small." Since S_0 is by hypothesis "small," $S_0 < S^*(1/\lambda)$ and the second part of Proposition 1 is of relevance (see Figure 2). The date of innovation is $1/\lambda$. Clearly $D_t = D(p_0 e^{rt})$ for $0 \leq t \leq 1/\lambda$, and so

$$(17) \qquad \bar{\eta} r S_0 = D(p_0) - D(p_{1/\lambda}).$$

From equations (16) and (17) it is immediate that

$$(18) \qquad \lim_{S_0 \to 0} D(p_0^u)/D(p_0) = (r + \lambda) \lim_{S_0 \to 0} [1 - D(p_0 e^{r/\lambda})/D(p_0)]/r.$$

It remains to calculate $D(p_0 e^{r/\lambda})/D(p_0)$ when $S_0 \to 0$. Now $p_0 \to \infty$ when $S_0 \to 0$. Thus define $\hat{\eta} \equiv \lim_{p \to \infty} \eta(p)$. From equation (10) it is then immediate that $D(p_0 e^{r/\lambda})/D(p_0) \to e^{-r\hat{\eta}/\lambda}$ as $S_0 \to 0$. Equation (18) therefore reduces to

$$(19) \qquad \lim_{S_0 \to 0} [D(p_0^u)/D(p_0)] = (r + \lambda)(1 - e^{-r\hat{\eta}/\lambda})/r.$$

Therefore the question is whether the right-hand side of equation (19) is greater or less than unity. If the former, then uncertainty provides an argument for a more rapid extraction policy. If the latter, then it provides an argument for a more conservative policy. Now, the right-hand side of (19) is monotonically increasing in $\hat{\eta}$ and tends to zero as $\hat{\eta} \to 0$. Moreover it is a trivial matter to check that the right-hand side exceeds unity if $\hat{\eta} = 1$. It follows that there exists an $\eta^*(<1)$ such that if $\hat{\eta} < \eta^*$, the right-hand side of (19) is less than unity and if $\hat{\eta} > \eta^*$ it is greater than unity. We can therefore summarize the findings by way of the following proposition.

PROPOSITION 3: *If $S_0 \geq S^*(1/\lambda)$, uncertainty in the date of invention of the new technology leads to greater conservation of the resource. If S_0 is "small", and if demand elasticity at high enough prices exceeds a critical value $\eta^*(<1)$, it provides an argument for a more rapid extraction policy.*

This proposition appeared surprising to us at first. After all, it is exactly when there is great scarcity of resource that society needs to be most careful about its use. Unguided intuition thus suggests that if the resource base in small uncertainty about future availability of a backstop technology would lead to greater conservationism. What this overlooks is that when S_0 is small the cost of reducing the resource flow is also greater. To see this consider first the case where S_0 is large. Then small variations in the invention date can make no difference, for Proposition 1 tells us that innovation occurs some time *after* invention. For large variations there is an asymmetry between increases and decreases in the arrival date: the latter has no effect, but the former lowers utility and increases the marginal utility of having reserves. Hence the desirability of greater conservation when reserves are large.

For small reserves (relative to the expected invention date $(1/\lambda)$) both increases and decreases in the invention date have effects. Interpreting p_0 as the marginal value of a unit of resource consumed at $t = 0$ it is simple to confirm that it is an increasing convex function of the invention date T^*, and that for large T^* (small S_0) the function's second derivative is bounded away from zero. On the other hand the present-value of a unit of the resource at the invention date declines with T^* and the second derivative of this function tends to zero for large T^* if demand is not too inelastic at high enough prices. In the absence of uncertainty, equilibrium is attained when a unit of the resource consumed at $t = 0$ has equal value to one postponed until T^* (see Figure 6). It is clear then that randomness in T^* increases the expected marginal value of present consumption and leaves relatively unaffected the marginal value of postponed consumption if $\hat{\eta} > \eta^*$. It is this that provides the motivation for greater profligacy in the presence of uncertainty. In general, as Figure 6 makes clear, randomness in T^* increases the expected marginal value of postponed consumption as well. The relative magnitudes of the increase in expected marginal value of present consumption and that in postponed consumption due to randomness in T^* are given by the right-hand side of equation (19).

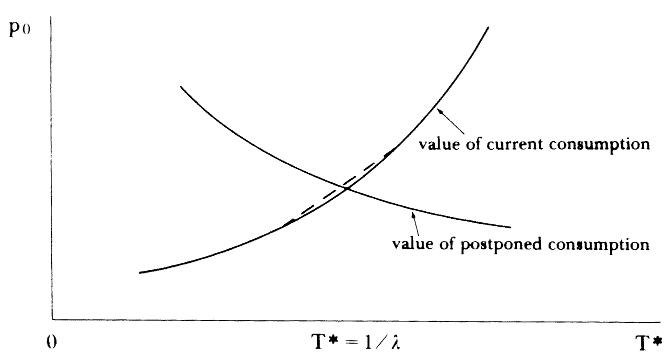

FIGURE 6

Proposition 3 has provided us with a general statement regarding the nature of the bias in initial rates of extraction for the case when $S_0 \geq S^*(1/\lambda)$ (see Figure 5), viz. that $A'A$ lies strictly above $B''B'$. We now proceed to see whether we can say something more precise about the number of intersections that $A'A$ has with $B'B$ in Figure 5. Thus consider the case where $S_0 < S^*(1/\lambda)$. At an intersection of $A'A$ and $B'B$ (i.e. $p_0^u = p_0$) we note that on using equation (17) in equations (9) and (11)

$$(20) \qquad \frac{dp_0^u/dS_0}{dp_0/dS_0} = \frac{r\bar{\eta}S_0}{D(p_0)} + \frac{\lambda(1-\hat{p}(S_0)/p_0)\bar{\eta}S_0}{D(p_0)}$$

$$= \{1 - D(p_{1/\lambda})/D(p_0)\}\left\{1 + \frac{\lambda}{r}(1-\hat{p}(S_0)/p_0)\right\}.$$

On differentiating (20) logarithmically with respect to S_0 we obtain

$$(21) \qquad \frac{d\ln(dp_0^u/dS_0/dp_0/dS_0)}{dS_0} = \frac{D(p_{1/\lambda})/D(p_0)}{1-D(p_{1/\lambda})/D(p_0)} \cdot \left[\frac{\eta(p_{1/\lambda})-\eta(p_0)}{\bar{\eta}S_0}\right] + \frac{\lambda}{r}\frac{\hat{p}(S_0)}{p_0}\left[\frac{1}{\bar{\bar{\eta}}S_0 + D(\bar{p})/r} - \frac{1}{\bar{\eta}S_0}\right] \Big/ \left\{1 + \frac{\lambda}{r}\left(1 - \frac{\hat{p}(S_0)}{p_0}\right)\right\}.$$

Now, if we can locate conditions under which the right-hand side of equation (21) is negative for all $S_0 < S^*(1/\lambda)$, we can conclude that under such conditions $A'A$ and $B'B$ intersect at most twice. Since $p_{1/\lambda} > p_0$, we know that $D(p_0) > D(p_{1/\lambda})$. Moreover, since $S_0 < S^*(1/\lambda)$, we known that $p_0 > \hat{p}(S_0)$ (Proposition 1). Therefore, if we can locate conditions on the demand function which ensure that both the expressions within square brackets in equation (21) are negative we shall have completed our task.

Consider first the class of iso-elastic demand functions, i.e. $\eta(p) = \eta$. For this class of cases the expression within the first pair of square brackets in the right-hand side of equation (21) is nil. Moreover when η is constant $\bar{\eta} = \bar{\bar{\eta}}$. Therefore the expression within the second pair of square brackets in the right-hand side of (21) is negative. Next consider the class of demand functions whose elasticities monotonically decline with price; i.e. $\eta'(p) < 0$. For this class of cases $\eta(p_{1/\lambda}) < \eta(p_0)$, and $\bar{\bar{\eta}} > \bar{\eta}$, and therefore each of the expressions within the two square brackets in the right-hand side of (21) is negative. Combining these two classes of cases we conclude that if $\eta'(p) \leq 0$,

$$d \ln \left\{\frac{dp_0^u/dS_0}{dp_0/dS_0}\right\} \Big/ dS_0 < 0.$$

Now, we have noted in Proposition 3 that there exists an $\eta^*(<1)$ such that if $\hat{\eta} \equiv \lim_{p\to\infty} \eta(p) > \eta^*$ then $p_0^u < p_0$ when $S_0 \to 0$. Thus, if $\hat{\eta} > \eta^*$, $A'A$ must lie below $B'B$ (Figure 5) for small enough S_0. But $A'A$ lies above $B''B'$. Therefore, if $\eta'(p) \leq 0$ and $\hat{\eta} \equiv \lim_{p\to\infty} \eta(p) > \eta^*$, then $A'A$ crosses $B'B$ precisely once (the case depicted in Figure 5). But if $\hat{\eta} < \eta^*$ then from equation (19) we note that $A'A$ lies above $B'B$ for small enough S_0. For this case we are unable to say whether $A'A$ crosses $B'B$ twice or whether it does not cross it at all and lies entirely above $B'B$. We can summarize this by way of the following proposition:

PROPOSITION 4: *If* $\eta'(p) \leq 0$, *there exists an* $\eta^*(<1)$ *such that*: (i) *if* $\lim_{p\to\infty} \eta(p) > \eta^*$ *then there exists a critical stock level* $\tilde{S}$, *such that uncertainty about the date of invention of the new technology leads to greater or less conservation of the resource depending on whether the initial stock*, S_0, *is greater or less than* $\tilde{S}$; (ii) *if* $\lim_{p\to\infty} \eta(p) < \eta^*$, *then either* (a) *uncertainty in the date of invention of the new technology leads to greater conservation irrespective of the size of the initial stock, or* (b) *there exist two critical values of the stock*, $\hat{S}$ *and* $\hat{\hat{S}}$, (*with* $\hat{S} < \hat{\hat{S}}$) *such that*

$$p_0^u > p_0 \quad \text{if} \quad S_0 < \hat{S} \quad \text{or} \quad S_0 > \hat{\hat{S}}, \quad \text{and}$$

$$p_0^u < p_0 \quad \text{if} \quad \hat{S}_0 < S_0 < \hat{\hat{S}}$$

(*i.e. for high or low initial stocks, uncertainty leads to greater conservation, but for middle size stocks it leads to profligacy*).

6. CONCLUDING REMARKS

Rothschild and Stiglitz [**10**] have provided a general methodology for analyzing the effect of uncertainty on optimum decisions when the decision variable itself is a finite dimensional vector. In this paper we have attempted to make a preliminary study of the manner in which we need to analyze the effect of uncertainty on optimum decisions when the decision variable is an infinite dimensional vector. The problem, in general, is unusually complicated, and we have been forced here to simplify a good deal.[20] Consequently, Propositions 3 and 4 need to be interpreted with caution. As we have postulated a Poisson process, we are unable to vary the variance without varying the mean, as we would ideally like to do. Nor is it obvious that the relevant comparison is with the *expected* date of invention (rather than some other statistic, e.g. the median). But the results reported here are, to us, suggestive. At any event, one of our goals was to see how uncertainty in future technology could be incorporated in a simple manner so as to display the basic characteristics of the transition from an exhaustible to an inexhaustible resource.

The basic results as regards this last objective on our part would appear to be those summarized in Propositions 1 and 2. For our purposes here it is worth emphasizing two features of these Propositions. First, they imply that if the date of invention of a substitute product is known in advance with certainty, the rate of extraction of an exhaustible natural resource ought to be chosen in such a manner as to deplete the stock at a known date in the future (Proposition 1), and that prior to exhaustion the resource price ought to obey equation (1). However, if the date

[20] For a somewhat related analysis, see Levhari and Mirman [**7**].

of invention is uncertain the rate of extraction ought to be chosen in such a manner that the economy possesses a positive stock so long as the invention has not occurred.[21] Moreover the resource price, prior to invention, ought to satisfy equation (2) (Proposition 2). But these characteristics imply that the rates of extraction prior to invention are different for the two cases. In particular, they imply that there is no certainty equivalent date of invention. However, equation (2) does imply that there are certainty equivalent *discount rates* for problem (6). Thus suppose we were to ignore the possibility of the invention occurring and instead were to assume that it will never occur. The planning problem would then consist of finding the optimal rate of depletion of the resource stock of size S_0 (i.e. the classic cake-eating problem). But now suppose that instead of using the constant rate, r, for discounting the flow of social surplus, $u(x_t)$, we were to use the variable rate given by the right-hand side of equation (2). Then obviously the pace of resource extraction along the solution of this cake-eating problem will coincide with the solution of problem (6) so long as the invention is not at hand. Increasing the discount rate so as to allow for uncertainty is a practice that has on occasion been advocated. What equation (2) describes is the precise manner in which this ought to be done and shows as well why the procedure is a complicated one. However, inequality (3) shows us within what bounds these effective discount rates must lie. It is only when $\hat{p}(S_t)/p_t$ is negligible that increasing the discount rate r by the conditional probability rate of success, λ_t, is a legitimate procedure. It was this last rule that was brought out by Yaari [**12**] in his analysis of a consumer with no bequest motive, planning in the face of an uncertain life time, and by Dasgupta and Heal [**2**] in their analysis of optimal capital accumulation and resource depletion in the face of uncertain future technology.

Of equal importance is the other feature we would wish to emphasize, that for the model at hand even in those cases where *invention* precedes resource exhaustion (first part of Proposition 1 and Proposition 2) technological *innovation* does not precede resource exhaustion. The model implies that exhaustion ought always to precede innovation. This would appear to be telling on the manner in which we often impute causality in economics. We have in the introduction referred to the timber famine in 16th Century England for an interpretation of which our analysis may be of relevance. The fact that innovation follows depletion does not provide one with reasons for supposing that a dwindling resource base is the cause of the innovation. "*Post hoc, ergo propter hoc*" is a fallacy that can only too easily be committed in the field of resource economics.

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[21] The assumption that $\lim_{x\to 0} f(x) = \infty$ is crucial for this result.

APPENDIX 1

We re-write problem (4) as

$$\max_{(x_t, y_t)} \int_0^\infty e^{-rt}(u(x_t+y_t)-\bar{p}y_t)\,dt$$

subject to the constraints

$$dS_t/dt = -x_t,$$

$$x_t,\ y_t,\ S_t \geq 0 \quad \text{all } t \geq 0; \quad y_t \geq 0, \quad \text{for} \quad 0 \leq t \leq T^* \quad \text{and} \quad S_0 \text{ given.}$$

To analyze the problem we may express the Hamiltonian as

(A.1) $$H = e^{-rt}(u(x_t+y_t)-\bar{p}y_t) - e^{-rt}p_t x_t + e^{-rt}\mu_t x_t + e^{-rt}\nu_t y_t + e^{-rt}\psi_t S_t,$$

where $e^{-rt}p_t$ is the auxiliary variable associated with the differential equation in the problem, and

(A.2) $$\left.\begin{aligned} e^{-rt}\mu_t \geq 0,\\ x_t \geq 0,\end{aligned}\right\}\text{complementary slackness (C-S)}$$

(A.3) $$\left.\begin{aligned} e^{-rt}\nu_t \geq 0,\\ y_t \geq 0,\end{aligned}\right\}\text{(C-S)}$$

(A.4) $$\left.\begin{aligned} e^{-rt}\psi_t \geq 0,\\ S_t \geq 0.\end{aligned}\right\}\text{(C-S)}$$

From (A.2) and (A.4) it is clear that $x_t > 0$ implies that $\mu_t = 0$ and therefore that $\psi_t = 0$. Define $z_t = x_t + y_t$. Then we know from (A.1) that

(A.5) $$u'(z_t) = p_t - \mu_t = \bar{p} - \nu_t.$$

Moreover, the auxiliary variable must satisfy the equation

(A.6) $$\frac{dp_t/dt}{p_t} = r - \psi_t/p_t.$$

Therefore, so long as $S_t > 0$, we must have $\psi_t = 0$ (A.4), and so equation (A.6) reduces to equation (1) in the text.

By assumption $\lim_{z\to 0} u'(z) = \infty$. Hence from (A.5) it is immediate that $z_t > 0$ for all $t \geq 0$. Now suppose first that $T^* = 0$ (i.e. the invention is at hand). Consider the policy described in the first part of Proposition 1 (See Figure 1). In particular, choose p_0 and $\hat{T}$ and y_t in such a manner that

(A.7) $$\int_0^{\hat{T}} D(p_0\, e^{rt})\,dt = S_0,$$

(A.8) $$p_0\, e^{r\hat{T}} = \bar{p},$$

and $u'(y_t) \equiv f(y_t) = \bar{p}$ for $t \geq \hat{T}$, and $x_t > 0$ and $y_t = 0$ for $0 \leq t \leq \hat{T}$. This candidate path satisfies the necessary conditions for optimality given above. Since $S_t = 0$ for $t \geq \hat{T}$, we note that $\lim_{t\to\infty} e^{-rt}p_t S_t = 0$. The Hamiltonian (A.1) is strictly concave in the control variables. The candidate path is therefore the unique optimum (see Arrow and Kurz [**1**, Chapter II]).

The optimum innovation date, $\hat{T}$, for the case just discussed ($T^* = 0$) is obtained from equations (A.7) and (A.8). It is immediate that $d\hat{T}/dS_0 > 0$. Now suppose $0 < T^* \leq \hat{T}$. It is then easy to verify that the path just analyzed is still the optimum path.

Finally suppose $T^* > \hat{T}$. Consider the policy described in the second part of Proposition 1 (see Figure 2). In particular, choose p_0 and y_t in such a way that

$$\int_0^{T^*} D(p_0\, e^{rt})\,dt = S_0,$$

and

$$f(y_t) = u'(y_t) = \bar{p} \qquad \text{for} \qquad t \geq T^*.$$

This policy also satisfies the necessary conditions for optimality and the transversality condition. It is therefore the unique optimum. We have therefore confirmed Proposition 1. The length of time during which the invention remains unused is max $(0, \hat{T} - T^*)$.

APPENDIX 2

Let us re-write the planning problem (6) as

$$\text{(A.9)} \qquad \underset{(x_t)}{\text{maximize}} \int_0^\infty e^{-rt}[u(x_t)(1-\Omega_t)+\Pi_t V(S_t)]\,dt$$

subject to $dS_t/dt = -x_t$,

$x_t, S_t \geq 0$ all $t \geq 0$ and S_0 given.

Suppose (A.9) has a solution. To analyze the problem let $e^{-rt}q_t$ denote the auxiliary variable and express the Hamiltonian as

$$\text{(A.10)} \qquad H = e^{-rt}[u(x_t)(1-\Omega_t)+\Pi_t V(S_t)] - e^{-rt}q_t x_t.$$

Since by hypothesis $\lim_{x\to 0} u'(x) = \infty$, and $\Omega_t < 1$ for all $t \geq 0$, we know in advance that $x_t > 0$ for all $t \geq 0$ along the solution of the above problem.

For x_t to be optimal at t it is immediate from (A.10) that

$$\text{(A.11)} \qquad u'(x_t)(1-\Omega_t) = q_t, \qquad t \geq 0.$$

Moreover the auxiliary variable must also satisfy the differential equation

$$\text{(A.12)} \qquad \frac{d}{dt}(e^{-rt}q_t) = -\partial H/\partial S_t = -e^{-rt}\Pi_t V'(S_t)$$

or

$$\dot{q}_t/q_t = r - \Pi_t \hat{p}(S_t)/q_t,$$

where

$$\hat{p}(S_t) = V'(S_t).$$

Now the spot price of the resource at t conditional on the invention not having occurred by then is of course $f(x_t) = u'(x_t) = q_t/(1-\Omega_t)$ (equation (A.11)). Define $p_t = u'(x_t) = q_t/(1-\Omega_t)$. Then clearly

$$\text{(A.13)} \qquad \dot{q}_t/q_t = \dot{p}_t/p_t - \Pi_t/(1-\Omega)$$

(since $\Pi_t = \dot{\Omega}_t$). But $\Pi_t/(1-\Omega_t) = \lambda_t$. If we now use (A.12) in (A.13) we obtain the equation

$$\text{(A.14)} \qquad \dot{p}_t/p_t = r + \lambda_t(1-\hat{p}(S_t)/p_t),$$

which is none other than equation (2) in the text.

We shall not aim at providing a general existence theorem for problem (A.9). Rather, we shall establish below that it has a unique solution for the case where the stochastic process is a Poisson one, the case analyzed in Section 4. But we note first that the integrand in (A.9) is a continuous function of time and strictly concave as a function of the control variable, x_t, and the state variable, S_t. Therefore, by Proposition 8 in Arrow and Kurz [**1**, Chapter II], a feasible policy is (uniquely) optimal if it satisfies equation (A.14) and the transversality conditions

$$\text{(A.15)} \qquad \lim_{t\to\infty} e^{-rt}(1-\Omega_t)p_t \geq 0 \qquad \text{and} \qquad \lim_{t\to\infty}(1-\Omega_t)p_t e^{-rt}S_t = 0.$$

We can now confirm that a unique optimum exists if the stochastic process is a Poisson one. Thus suppose that $\Pi_t = \lambda e^{-\lambda t}$ $(\lambda > 0)$. For this case the integral in (A.9) reduces to

$$\text{(A.16)} \qquad \int_0^\infty e^{-(r+\lambda)t}[u(x_t)+\lambda V(S_t)]\,dt.$$

For the Poisson case (A.15) becomes

$$\text{(A.17)} \qquad \lim_{t\to\infty} e^{-(r+\lambda)t}p_t \geq 0 \qquad \text{and} \qquad \lim_{t\to\infty} e^{-(r+\lambda)t}p_t S_t = 0.$$

Now consider the phase path $A'A$ in Figure 5. By construction, $S_t > 0$ for $t \geq 0$ and $\lim_{t\to\infty} S_t = 0$, and of course (A.14) is satisfied along it. But equation (A.14) tells us that $r \leq \dot{p}_t/p_t \leq r+\lambda$. Thus (A.17) is satisfied along $A'A$.

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